SOLUTIONS OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From R. J. Adcock, 299, 307, 308; Prof. L. G. Barbour, 296; Prof. W. P. Casey, 296, 299, 300, 301, 302; Alex. S. Christie, 301, 302; Prof. E. J. Edmunds, 296; George Eastwood, 298, 300; Dr. Wm. Hillhouse, 296, 300; W. E. Heal, 300, 302, 303, 305, 306; Prof. W. W. Johnson, 306; Chas. H. Kummell, 303, 308; Prof. J. H. Kershner, 296, 298, 299, 300, 301, 303; Prof. D. J. Mc Adam, 296; P. Richardson, 296, 300; T. P. Stowell, 298; Prof. E. B. Seitz, 296, 300, 307; Prof. J. Scheffer, 296, 298, 299, 300, 301, 303, 304; E. P. Thompson, 296.

296. "Given $x^6+y^6=\frac{1}{2}\sqrt{2}$, $x^4+y^4=1$; to find x and y by quadratics."

SOLUTION BY PROF. L. G. BARBOUR, RICHMOND, KY.

Let $m+n = x^2$, $m-n = y^2$; then

$$m^3+3m^2n+3mn^2+n^3+m^3-3m^2n+3mn^2-n^3 = x^6+y^6 = \tfrac{1}{2}\sqrt{2}.$$

$$\therefore m^3+3mn^2 = \tfrac{1}{2}\,.\,\tfrac{1}{2}\sqrt{2} = \tfrac{1}{4}\sqrt{2}. \qquad (1)$$

Also $m^2+2mn+n^2+m^2-2mn+n^2 = x^4+y^4 = 1$; $\therefore m^2+n^2 = \frac{1}{2}$.

$\therefore n^2 = \frac{1}{2}-m^2 =$ by (1), $(\frac{1}{4}\sqrt{2}-m^3)\div 3m$, or $\frac{3}{2}m-3m^3 = \frac{1}{4}\sqrt{2}-m^3$.

$$\therefore 4m^3-3m = -\tfrac{1}{2}\sqrt{2},\ \therefore 4m^4-3m^2 = -\tfrac{1}{2}\sqrt{2}\,.\,m,$$

or

$$4m^4-m^2+\tfrac{1}{16} = 2m^2-\tfrac{1}{2}\sqrt{2}\,.\,m+\tfrac{1}{16};$$

$$\therefore 2m^2-\tfrac{1}{4} = m\sqrt{2}-\tfrac{1}{4};\ \therefore 2m = \sqrt{2},\text{ or } m = \tfrac{1}{2}\sqrt{2}.$$

Therefore $n^2 = 0$, and $x^4 = m^2 = \frac{1}{2}$; $\therefore x = \frac{1}{2}\sqrt[4]{2} = y$.

[This problem was also solved as above, with a slight difference in notation, by Professors Seitz and Mc Adam.]

297. No solution received.

298. "On the diameter AB produced, of a given circle, is a given point P. A chord CD is parallel to the diameter AB. Find the position of CD, by a geometrical construction, such that the angle CPD shall be the greatest possible."

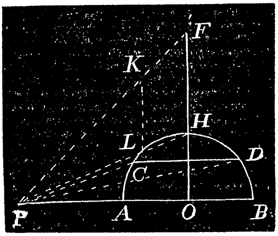
SOLUTION BY PROF. J. H. KERSHNER, MERCERSBURG, PA.

Represent the point D by (x, y), the origin being at the centre. The tan of $CPD = \tan(CPB-DPB) = (\tan CPB-\tan DPB)\div(1+\tan CPB\times \tan DPB)$

$$= \left(\frac{y}{a-x} - \frac{y}{a+x}\right) \div \left(1 - \frac{y^2}{a^2 - x^2}\right)$$

$$= \left(\frac{2xy}{a^2 - x^2 + y^2}\right) = \text{a maximum},$$

where $a = PO$.



Substituting $\sqrt{(r^2 - x^2)}$ for y, we find by the Diff. Calculus, the expression has a maximum for

$$x = \frac{r\sqrt{(a^2 - r^2)}}{a\sqrt{2}}.$$

To find x by geometrical construction, erect the perpendicular $OF = OP$, join PF and PH, lay off $FK = r$, draw KL, parallel to OF, meeting PH in L, and we shall have $HL = x =$ the abscissa of C or D.

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299. "Prove that the number of terms in an *expanded* polynomial of t terms is equal to the number of combinations of $(n+t-1)$ quantities taken $t-1$ at a time, where n is the exponent of the polynomial."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

The number of terms in the expansion of any polynomial, the exponent n, being a positive integer, is

$$\frac{t(t+1)(t+2)\ldots.(t+n-1)}{n!},$$

where t is the number of terms and n the exponent; which is evidently $=$ to the number of combinations of $(n+t-1)$ quantities taken $t-1$ at a time.

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

The expansion of $(a+b)^n$ has $n+1$ terms; that of $(a+b+c)^n$ $= a^n + na^{n-1}(b+c) + \frac{1}{2}n(n-1)a^{n-2}(b+c)^2 + \ldots. + na(b+c)^{n-1} + (b+c)^n$, which has the sum of $n+1$ terms of the series $1+2+3+4+\ldots+(n+1)$ $= \frac{1}{2}(n+2)(n+1)$ terms. The expansion of $(a+b+c+d)^n$ has the sum of $n+1$ terms of the series $1+3+6+\ldots+\frac{(n+1)(n+2)}{1.2} = \frac{(n+3)(n+2)(n+1)}{1.2.3}$ by differences. In like manner, the expansion of $(a+b+c+d+e)^n$ has $\frac{(n+4)(n+3)(n+2)(n+1)}{1.2.3.4}$ terms. Hence $\frac{(n+t-1)(n+t-2)\ldots(n+1)}{1.2.3\ldots(t-1)}$ which expresses the proposition that was to be proved.

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300. "In a plane trian. ABC are given, the perimeter $2s$ and the radii of the inscribed and circumscribed circles, r and R, to determine the triangle."

SOLUTION BY P. RICARDSON, BROOKLYN, N. Y.

Let the sides opposite the angles A, B, C, be a, b, c, then we have given

$$a + b + c = 2s. \tag{1}$$

Double the area is $ab \sin C = 2rs$, $\sin C = c \div 2R$; $\therefore$ by substitution we get

$$abc = 4Rrs. \tag{2}$$

The area is $\sqrt{[s(s-a)(s-b)(s-c)]} = rs$, or, by squaring and performing operations indicated in left hand member, we get

$$s^4-(a+b+c)s^3+(ab+ac+bc)s^2-abcs = r^2s^2,$$

and by substituting from (1) and (2) we get

$$[-s^4+(ab+ac+bc)s^2-4Rrs^2] = r^2s^2,$$

whence

$$ab+ac+bc = r^2+s^2+4Rr. \tag{3}$$

Equations (1), (2) and (3) show that a, b, c, must be the roots of the cubic equation $\quad x^3-2sx^2+(r^2+s^2+4Rr)x-4Rrs = 0.$

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301. "Determine the angles of a plane triangle from the following relations.

$$\text{(I)} \begin{cases} \tan^2 A+\tan^2 B+\tan^2 C = m, & (1) \\ \tan^4 A+\tan^4 B+\tan^4 C = n. & (2) \end{cases}$$

$$\text{(II)} \begin{cases} \tan^2\tfrac{1}{2}A+\tan^2\tfrac{1}{2}B+\tan^2\tfrac{1}{2}C = m, & (1) \\ \tan^4\tfrac{1}{2}A+\tan^4\tfrac{1}{2}B+\tan^4\tfrac{1}{2}C = n. & (2) \end{cases}$$

$$\text{(III)} \begin{cases} \tan\tfrac{1}{2}A+\tan\tfrac{1}{2}B+\tan\tfrac{1}{2}C = m, & (1) \\ \tan\tfrac{1}{2}A\times\tan\tfrac{1}{2}B\times\tan\tfrac{1}{2}C = n.\text{"} & (2) \end{cases}$$

SOLUTION BY ALEX. S. CHRISTIE, U. S. C. SURV., WASH., D. C.

Let the sought tangents in each case be the roots a, b, c of the cubic

$$x^3+px^2+qx+r = 0.$$

Then by well known forms in the theory of equations and trigonometry (say Faa' de Bruno's Forms, Binaires, p. 27, and Chauvenet's Trig., p. 62) we have for determining p, q, r,

$$\text{(I)} \quad \begin{cases} m = a^2+b^2+c^2 = p^2-2q & (1) \\ n = a^4+b^4+c^4 = p^4-4p^2q+4pr+2q^2 & (2) \\ 0 = p-r & (3) \end{cases}$$

(II) $\quad$ (1) and (2) with $0 = 1-q$

$$\text{(III)} \quad \begin{cases} m = -p \\ n = -r \\ 0 = 1-q \end{cases}$$

Whence, for the three cases in order

$$x^3+x^2\sqrt{(m+2q)}+xq+\sqrt{(m+2q)}=0,$$

where $$q=\tfrac{1}{2}\{4\pm\sqrt{[16-2(n-4m-m^2)]}\};$$

$$x^3+x^2\sqrt{(m+2)}+x+\frac{n-m^2+2}{4\sqrt{(m+2)}}=0,$$

$$x^3-mx^2+x-n=0.$$

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302. "Given n points in space, to construct a polygon of n sides, having its sides bisected by the n given points, and determine the No. of solutions."

SOLUTION BY ALEX. S. CHRISTIE.

With any point in space as origin, let ρ_n, ρ_{n-1}, ... ρ_1 be the quaternion vectors to the vertices of the required polygon, and α_n, α_{n-1}, ... α_1 the corresponding intermediate vectors to the given points.

Then
$$\begin{aligned}\rho_n &= \alpha_n + \alpha_n - \rho_{n-1} = 2\alpha_n - \rho_{n-1}\\ &= 2\alpha_n - 2\alpha_{n-1} + \rho_{n-2}\\ &= 2\alpha_n - 2\alpha_{n-1} + \alpha_{n-2} - \rho_{n-3}\\ &\ldots\ldots\ldots\\ &= 2\alpha_n - 2\alpha_{n-1} + 2\alpha_{n-2} - 2\alpha_{n-3} + \ldots + (-)^{n-1}2\alpha_1 + (-)^n\rho_n.\end{aligned}$$

When n is even $\quad 0 = \alpha_n - \alpha_{n-1} + \alpha_{n-2} - \alpha_{n-3} + \ldots + \alpha_2 - \alpha_1.$ (1)

" n is odd $\quad \rho_n = \alpha_n - \alpha_{n-1} + \alpha_{n-2} - \alpha_{n-3} + \ldots - \alpha_2 + \alpha_1.$ (2)

Hence, when n is even, there are an infinite number of solutions, or none, according as the geometrical condition (1) is, or is not fulfilled; and when n is odd, (2) gives the construction for any vertex. It is obvious that there are, in this case, $\frac{1}{2}(n-1)!$ different solutions.

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303. "Required the length of one branch of a curtate cycloid; the generating circle having a radius of 4 inches and the generating point a radius of 8 inches."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Denote the radius of the generating circle by $r=4$ inches and that of the generating point by $R=8$ inches. Suppose the rolling of the generating circle begins when the radius of the generating point has a position C_0P_0 vertically downward. Let θ be the angle of turning from this position, then we have, taking A for origin (see Fig. on next page),

$$x = AM = AN - MN = r\theta - R\sin\theta, \qquad (1)$$

$$y = PM = CN + PQ = r - R\cos\theta. \qquad (2)$$

We have then $dx=(r-R\cos\theta)d\theta$; $dy=R\sin\theta\, d\theta$.

Therefore $ds=d\theta\sqrt{[R^2-2Rr\cos\theta+r^2]}=d\theta\sqrt{[(R+r)^2-4Rr\cos^2\frac{1}{2}\theta]}$. (3)

We have then the whole branch

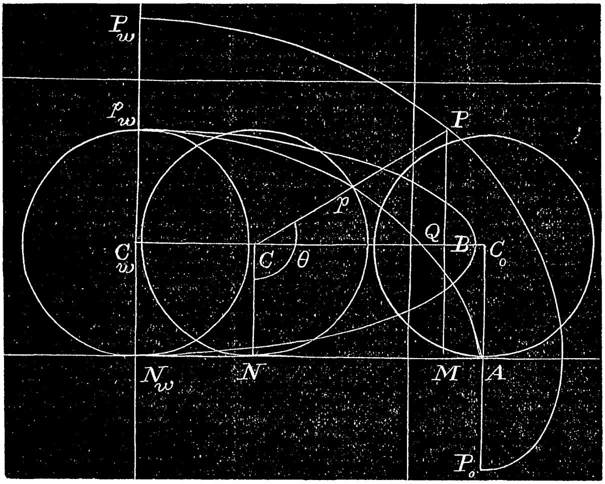
$$s_{2\pi}-s_0=\int_0^{2\pi} d\theta\,\sqrt{[(R+r)^2-4Rr\cos^2\tfrac{1}{2}\theta]}. \tag{4}$$

Place $\theta=\pi-2\varphi$ then

$$s_{2\pi}-s_0=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2d\varphi\,\sqrt{[(R+r)^2-4Rr\sin^2\theta]}$$

$$=4\int_0^{\frac{\pi}{2}} d\varphi\,\sqrt{[(R+r)^2\cos^2\varphi+(R-r)^2\sin^2\varphi]}, \tag{5}$$

$=4(R+r)\,E_0^{\frac{1}{2}\pi}\,\dfrac{2\sqrt{Rr}}{R+r}$ equal perimeter of ellipse $(R+r,\ R-r)$.



Place $R=8$ and $r=4$, then comparing (5) with (28), Vol. V, page 98, we have $a=48$ and $b=16$.

Having computed $a_1, b_1, a_2, b_2 \ldots$ by (15), (17), Vol. V, p. 19, we have by (36′) since $a_4=b_4$ for 7 pla's

$$s_{2\pi}-s_0=\pi\left(2^2b_4-2\frac{b_3^2}{b_4}-\frac{b_2^2}{b_4}-\frac{1}{2}\frac{b_1^2}{b_4}\right). \tag{6}$$

The computation is as follows:

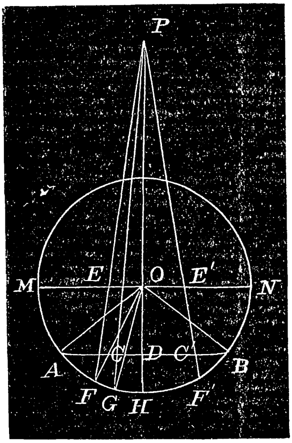
a	$=48$	$\log b$	$=1.2041200$	$\log 2^2$	$=0.6020600$
b	$=16$	$\log a$	$=1.6812412$	$\log b_4$	$=1.4752034$
a_1	$=32$	$\log b_1$	$=1.4426806$	$\log 2^2b_4$	$=2.0772634$
b_1	$=27.71281$	$\log a_1$	$=1.5051500$	$\log 2$	$=0.3010300$
a_2	$=29.95641$	$\log b_2$	$=1.4739153$	$\log b_3^2\div b_4$	$=1.4752016$
b_2	$=29.77935$	$\log a_2$	$=1.4764897$	$\log 2b_3^2\div b_4$	$=1.7762316$
a_3	$=29.86788$	$\log b_3$	$=1.4752025$	$\log b_2^2\div b_4$	$=1.4726272$
		$\log a_3$	$=1.4752044$	$\log \frac{1}{2}$	$=9.6989700$
		$\log b_4$	$=1.4752034$	$\log b_1^2\div b_4$	$=1.4101578$
				$\log \frac{1}{2}b_1^2\div b_4$	$=1.1091278$

2^2b_4	$=119.47124$	$\log \text{Sum}$	$=1.2352268$
$-2b_3^2\div b_4$	$=-\ 59.73537$	$\log\pi$	$=0.4971499$
$-\ b_2^2\div b_4$	$=-\ 29.69116$	$\log(s_{2\pi}-s_0)$	$=1.7323767$
$-\frac{1}{2}b_1^2\div b_4$	$=-\ 12.85665$		
Sum $=$	17.18806	$s_{2\pi}-s_0$	$=53.9976$ inches.

304. "Find the limit of error in the following method of trisecting an arc approximately: Trisect the chord of the arc, also the diameter parallel to it, and the correspondiag semicircumference. Join the corresponding points of trisection of the semicircumference and the diameter, producing the lines till they intersect. From the point thus found pass lines through the points of trisection of the chord until they meet the arc."

SOLUTION BY PROF. J. SCHEFFER, MERCERSBURG, PA.

Let E, E' represent the points of trisection of the diameter MN; C, C', those of the chord AB, and F, F', those of the semicircle.



Draw the perpendicular PD, and join the centre O with F and G. Denote the angle AOD by a, the angle GOH by φ, and the radius by r.

We now easily find $OE = \frac{1}{3}r$ $CD = \frac{1}{3}\sin a$.

In the triangle EOF, we have $EF^2 = EO^2 + OF^2 - 2EO\,.\,OF\cos 60°$; $\therefore EF = \frac{1}{3}r\sqrt{7}$; $\cos FEO = (FE^2 + EO^2 - OF^2) \div (2EF\,.\,EO) = -\frac{1}{14}\sqrt{7}$; $\therefore OEP = \frac{1}{14}\sqrt{7}$, and $\sin OEP = \frac{3}{14} \times \sqrt{21}$; therefore $\tan OEP = 3\sqrt{3}$. But $PO = \frac{1}{3}r \times \tan OEP = r\sqrt{3}$. Since $OD = r\cos a$, we have $PD = r(\cos a + \sqrt{3})$.

We have $\tan GPO = CD \div PD = \frac{1}{3}[\sin a \div (\cos a + \sqrt{3})]$; therefore

$$\sin GPO = \frac{\sin a}{\sqrt{[\sin^2 a + 9(\cos a + \sqrt{3})^2]}};\ \cos GPO = \frac{3(\cos a + \sqrt{3})}{\sqrt{[\sin^2 a + 9(\cos a + \sqrt{3})^2]}}$$

In the triangle GPO, we have $OG : OP = \sin GPO : \sin PGO$; $\therefore$

$$\sin PGO = \frac{\sqrt{3}.\sin a}{\sqrt{[\sin^2 a + 9(\cos a + \sqrt{3})^2]}};\ \cos PGO = \frac{\sqrt{[9(\cos a + \sqrt{3})^2 - 2\sin^2 a]}}{\sqrt{[\sin^2 a + 9(\cos a + \sqrt{3})^2]}}$$

But $\sin\varphi = \sin(GPO + PGO)$, therefore

$$\sin\varphi = \frac{\sin a\sqrt{[9(\cos a + \sqrt{3})^2 - 2\sin^2 a]} + 3\sqrt{3}.\sin a(\cos a + \sqrt{3})}{\sin^2 a + 9(\cos a + \sqrt{3})^2}.$$

With regard to the trisection of an arc this formula gives a correct result only for an arc of 180°. For other arcs, I find, if $a = 45°$, $\varphi = 15°\ 6'$; if $a = 30°$, $\varphi = 10°\ 2'$, &c.

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305. "Find the equation of the curve, rectangular co-ordinates (x, y) in which the length $s = \frac{ny}{x} = \int_0^x dx \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$, n being a constant."

SOLUTION BY W. E. HEAL, WHEELING, INDIANA.

Let the indefinite integral

$$\int dx\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}=\varphi(x),\text{ then }\int_0^x dx\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}=\varphi(x)\text{-}\varphi(0)=\frac{ny}{x}$$

Differentiating the above eq'n, (1), remembering that $\varphi(0)$ is constant,

$$dx\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}=\frac{n(xdy-ydx)}{x^2}. \qquad (2)$$

Put $dy\div dx=p$, and (2) becomes $x^2\sqrt{[(1+p^2)]}=npx-ny$. (3)

Differentiating (3) we have

$$\frac{xpdp}{\sqrt{(1+p^2)}}+2dx\sqrt{(1+p^2)}=ndp. \qquad (4)$$

Divide by $2(1+p^2)^{\frac{1}{4}}$; $dx(1+p^2)^{\frac{1}{4}}+xpdp\div(1+p^2)^{\frac{3}{4}}=ndp\div2(1+p^2)^{\frac{3}{4}}$. (5)

Integrating (5) and determining the constant $C=0$, we have

$x(1+p^2)^{\frac{1}{4}}=\frac{n}{2}\int\frac{dp}{(1+p^2)^{\frac{3}{4}}}$, (6). Let $p=\frac{2^{\frac{1}{2}}v(1-\frac{1}{2}v^2)^{\frac{1}{2}}}{1-v^2}$, then (6) becomes

$$\frac{x}{(1-v^2)^{\frac{1}{2}}}=\frac{n}{2^{\frac{1}{2}}}\int\frac{dv}{\sqrt{[(1-v^2)^3(1-\frac{1}{2}v^2)]}}=2^{\frac{1}{2}}n\left[\frac{v(1-\frac{1}{2}v^2)^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}}\right.$$

$$\left.+\frac{1}{2}\int\frac{dv}{(1-v^2)^{\frac{1}{2}}(1-\frac{1}{2}v^2)^{\frac{1}{2}}}-\int\frac{(1-\frac{1}{2}v^2)^{\frac{1}{2}}dv}{(1-v^2)^{\frac{1}{2}}}\right]=2^{\frac{1}{2}}n\left[\frac{v(1-\frac{1}{2}v^2)^{\frac{1}{2}}}{(1-v^2)^{\frac{1}{2}}}+\frac{1}{2}F(\tfrac{1}{2}2^{\frac{1}{2}},v)\right.$$

$\left.-E(\frac{1}{2}2^{\frac{1}{2}},v)\right]$ (7), where F and E are elliptic funct's of the 1st and 2nd ord's.

But $v=\frac{[(1+p^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}}{(1+p^2)^{\frac{1}{2}}}$, and from (3) $p=\frac{n^2y+x[n^2(x^2+y^2)-x^4]}{x(n^2-x^2)}$.

Substituting these values in (7) we have the equation of the curve.

[The solution of 306, by Prof. Johnson, of 307, by Prof. Seitz, and of 308, by Mr. Adcock, will be published in No. 4.]

PROBLEMS.

309. *By Prof. Beman.*—In a given circle, find the vertices of the inscribed square, pentagon, octagon, and decagon by using the dividers alone.

310. *By Prof. Edmunds.*—Required the locus of vertices of a right angled spherical triangle whose legs pass through two fixed points given on the surface of the sphere.

311. *By Prof. Casey.*—A uniform circular plate is placed with its centre upon a prop, to find at what points on its circumference three given weights p, q, r must be attached that it may remain at rest in a horizontal position.

312. *By Prof. Scheffer.*—To find the area of the loop of the curve $y^3+x^2y-axy+bx^2=0$.